Appendix B: Introduction to Circles

TOPICS

1. Standard form of a circle
2. Graphing circles

Topic 1: Standard Form of a Circle

The focus of this chapter so far has been on equations that define lines in $\mathbb{R}^2$. But as you no doubt suspect, linear equations are just the beginning. Much of the rest of this text will deal, directly or indirectly, with more interesting graphs in the plane. As a preview of the discussion to come, we will close out this chapter with a brief introduction to a particular class of nonlinear equations.

Circles in $\mathbb{R}^2$ can be visualized, graphed, and described mathematically. To begin with, note that two pieces of information are all we need to completely characterize a particular circle: the circle's center, and the circle's radius. To be specific, suppose (h, k) is the ordered pair corresponding to the circle's center, and suppose the radius is given by the positive real number r. Our goal is to develop an equation in the two variables x and y so that every solution (x, y) of the equation corresponds to a point on the circle.

The main tool that we need to achieve this goal is the distance formula derived in Section 3.1. Since every point (x, y) on the circle is a distance r from the circle's center (h, k), that formula tells us that

$$r = \sqrt{(x-h)^2 + (y-k)^2}.$$

This form of the equation for a circle is in many ways the most natural, but such equations are commonly presented in the radical-free form that results from squaring both sides:

$$r^2 = (x-h)^2 + (y-k)^2.$$

Standard Form of a Circle

The **standard form** of the equation for a circle of radius r and center (h, k) is

$$(x-h)^2+(y-k)^2=r^2.$$

example 1

Find the standard form of the equation for the circle with radius 3 and center $(-2, 7)$.

Solution:

We are given $h = -2$, $k = 7$, and $r = 3$, so the equation is determined to be

$$(x-(-2))^2+(y-7)^2=3^2.$$

This is better written as

$$(x+2)^2+(y-7)^2=9.$$

example 2

Find the standard form of the equation for each of the following circles:

a. A circle with a diameter whose endpoints are $(-4, -1)$ and $(2, 5)$.

b. A circle which is tangent to the line $x = -1$ and whose center is $(3, 5)$.

Solutions:

a. The midpoint of a diameter of a circle is the circle's center, so the first step is to use the Midpoint Formula as follows:

$$(h, k)=\left(\frac{-4+2}{2}, \frac{-1+5}{2}\right)=(-1, 2).$$

The distance from either diameter endpoint to the center defines the circle's radius. Since we ultimately will want r^2, we can use a slight variation of the Distance Formula to determine

$$r^2=(-4-(-1))^2+(-1-2)^2=9+9=18.$$

Thus, the equation we seek is

$$(x+1)^2+(y-2)^2=18.$$

Note that $(-4, -1)$ and $(2, 5)$, two points that by definition are on the circle, both satisfy this equation.

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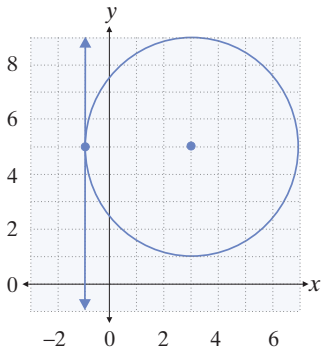
b. The word tangent in this context means that the circle just touches the line $x = -1$. In fact, by referring to a sketch, we can see that the described circle must touch the vertical line $x = -1$ at the point $(-1, 5)$. The distance between the center $(3, 5)$ and the point $(-1, 5)$ must then correspond to the radius of the circle, giving us $r = 4$. At this point, we are ready to construct the equation of the circle:

$$(x - 3)^2 + (y - 5)^2 = 16.$$



Topic 2: Graphing Circles

We will often need to reverse the process illustrated in the first two examples. That is, given an equation for a circle, we will need to determine the circle's center and radius and, possibly, graph the circle. If the equation is given in standard form, this is very easily accomplished.

example 3

Sketch the graph of the circle defined by

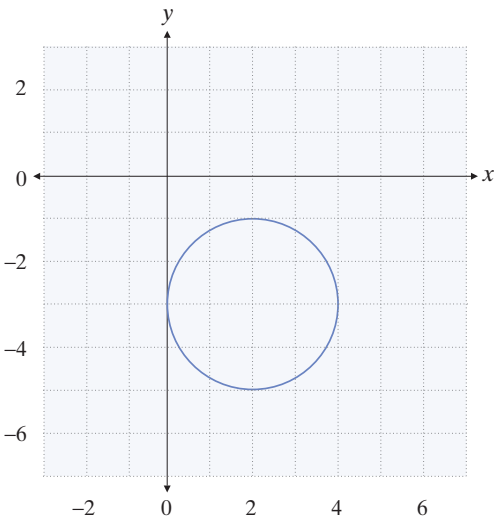
$$(x-2)^2 + (y+3)^2 = 4.$$

Solution:

The only preliminary step required is to slightly rewrite the equation in the form

$$(x-2)^2 + (y-(-3))^2 = 2^2.$$

From this, we see that $(h, k) = (2, -3)$ and that $r = 2$. The graph of the equation is thus



Typically, though, the equation for a circle will not be given to us in quite so neat a fashion. We may have to resort to a small amount of algebraic manipulation in order to determine that a given equation describes a circle and to determine the specifics of that circle. Fortunately, the algebraic technique of *completing the square* (Section 1.7) is usually all that is required.

example 4

Sketch the graph of the equation

$$x^2 + y^2 + 8x - 2y = -1.$$

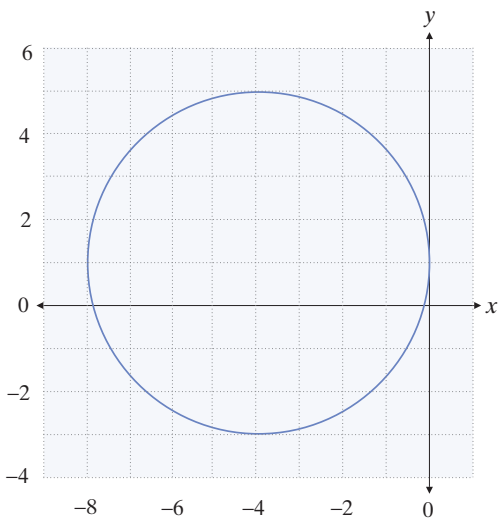
Solution:

We need to complete the square in the variable x and the variable y, and we do so as follows:

$$\left(x^2 + 8x + 16\right) + \left(y^2 - 2y + 1\right) = -1 + 16 + 1 \qquad \text{Add 16 and 1 to both sides.}$$

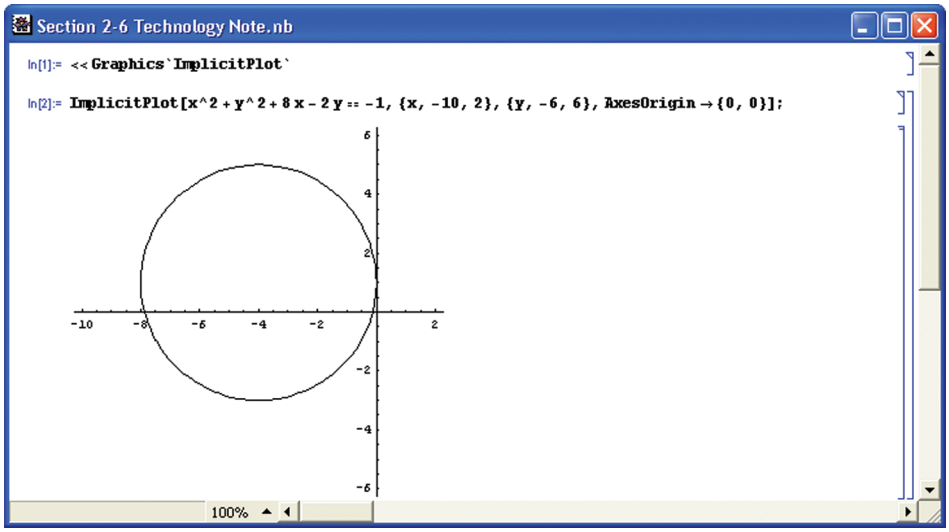
$$\left(x + 4\right)^2 + \left(y - 1\right)^2 = 16.$$

We now see that the equation does indeed describe a circle, and that the center of the circle is $(-4, 1)$ and the radius is 4. The graph appears below.



technology note

Graphing calculators and computer algebra systems, such as *Mathematica*, can be used to sketch accurate graphs of circles. To use *Mathematica* to do so, the command **<<Graphics`ImplicitPlot`** must be executed. This command allows *Mathematica* to graph equations, and needs to be executed just once per *Mathematica* session. The command used to then graph an equation is **ImplicitPlot**, and the details of its use are shown below.



exercises

Find the standard form of the equation for the circle. See Examples 1 and 2.

1. Center $(-4, -3)$; Radius 5

2. Center $(0, 0)$; Radius $\sqrt{6}$

3. Center $(7, -9)$; Radius 3

4. Center $\left(-\sqrt{4}, 2\right)$; Radius 2

5. Center at origin; Radius $\sqrt{9}$

6. Center $(6, 3)$; Radius 8

7. Center $\left(\sqrt{5}, \sqrt{3}\right)$; Radius 4

8. Center $\left(\frac{5}{3}, \frac{8}{5}\right)$; Radius $\sqrt{8}$

9. Center $(7, 2)$; tangent to the x-axis

10. Center at $(3, 3)$; tangent to the x-axis

11. Center at $(-3, 8)$; passes through $(-4, 9)$

12. Center at $(0, 0)$; passes through $(2, 10)$

13. Center $(4, 8)$; passes through $(1, 9)$

14. Center $(12, -4)$; passes through $(-9, 5)$

15. Center at the origin; passes through $(6, -7)$

16. Center $(13, -2)$; passes through $(8, -3)$

17. Endpoints of a diameter are $(-8, 6)$ and $(1, 11)$

18. Endpoints of a diameter are $(5, 3)$ and $(8, -3)$

19. Endpoints of a diameter are $(-7, -4)$ and $(-5, 7)$

20. Endpoints of a diameter are $(2, 3)$ and $(7, 4)$

21. Endpoints of a diameter are $(0, 0)$ and $(-13, -14)$

22. Endpoints of a diameter are $(4, 10)$ and $(0, 3)$

23. Endpoints of a diameter are $(0, 6)$ and $(8, 0)$

24. Endpoints of a diameter are $(6, 9)$ and $(4, 9)$

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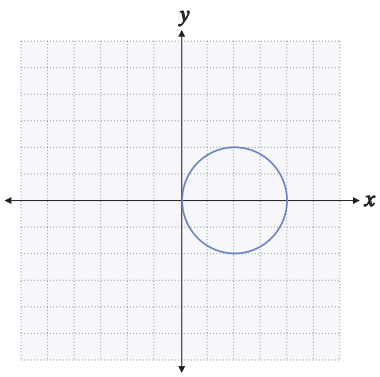

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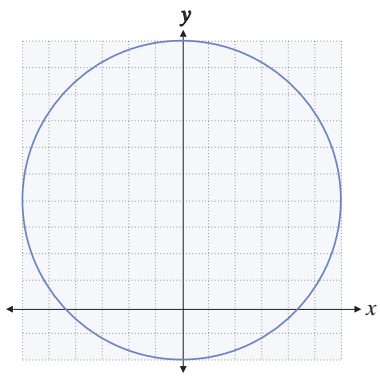

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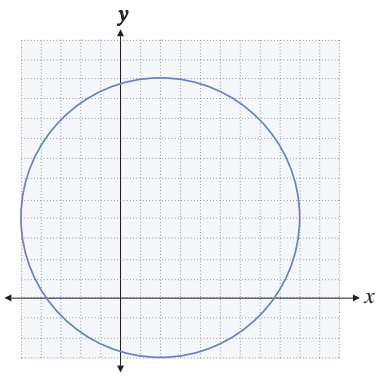

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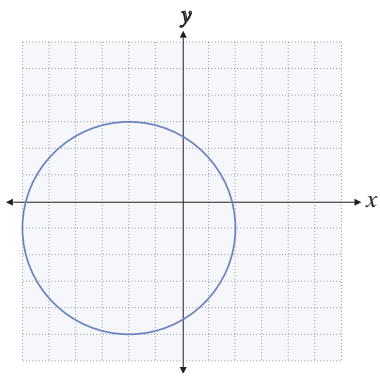

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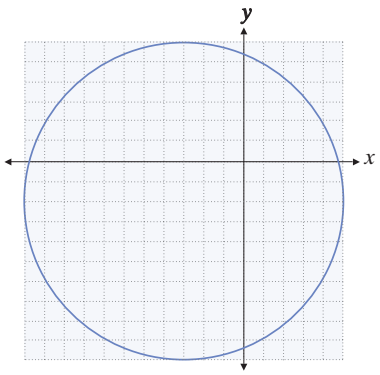

Sketch a graph of the equation. Then use a graphing calculator or computer to graph the circle. See Examples 3 and 4.

30. $x^2 + y^2 = 25$

31. $x^2 + y^2 = 36$

32. $x^2 + (y - 3)^2 = 16$

33. $x^2 + (y - 8)^2 = 9$

34. $(x + 2)^2 + y^2 = 169$

35. $(x - 8)^2 + y^2 = 8$

36. $(x - 9)^2 + (y - 4)^2 = 49$

37. $(x + 5)^2 + (y + 4)^2 = 4$

38. $(x + 2)^2 + (y - 7)^2 = 64$

39. $(x - 5)^2 + (y + 5)^2 = 5$

40. $x^2 + y^2 - 2x + 10y + 1 = 0$

41. $x^2 + y^2 - 4x + 4y - 8 = 0$

42. $x^2 + y^2 + 6x + 5 = 0$

43. $x^2 + y^2 + 10y + 9 = 0$

44. $x^2 + y^2 - x - y = 2$

45. $x^2 + y^2 + 6y - 2x = -2$

46. $(x - 5)^2 + y^2 = 225$

47. $4x^2 + 4y^2 = 256$

48. $(x - 3)^2 + (y + 2)^2 = 81$

49. $x^2 + y^2 - 6x + 4y - 3 = 0$

50. $(x + 2)^2 + (y - 1)^2 = 16$

51. $(x - 1)^2 + y^2 = 9$

52. $x^2 + (y + 2)^2 = 49$

53. $x^2 + y^2 - 4x + 8y - 16 = 0$

54. $x^2 + y^2 + 8x = 9$

55. $4x^2 + 4y^2 - 24x + 24y = 28$